# SECTION 15.7: EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

**RECALL:** Suppose $f$ is defined on an interval $I$.

- The value $f(c)$ is called a **local maximum** if $f(c) \geq f(x)$ for all $x$ in some open interval in $I$ containing $c$.
- The value $f(c)$ is called a **local minimum** if $f(c) \leq f(x)$ for all $x$ in some open interval in $I$ containing $c$.

Geometrically, the 'local' extrema of a function correspond to points which are the highest or lowest points when we zoom in near a point and compare function values on either side of that point.

**FERMAT'S THEOREM:** If $f(c)$ is a local extreme value of $f$, then $f'(c) = 0$ or $f'(c)$ does not exist.

**DEFINITION:** A value $c$ in the domain of $f$ is a **critical number** if $f'(c) = 0$ or $f'(c)$ does not exist.

**THE SECOND DERIVATIVE TEST FOR LOCAL EXTREMA:**
Suppose $f$ is continuous and $f'(c) = 0$.

- If $f''(c) > 0$ then $f$ has a local minimum at $x = c$.
- If $f''(c) < 0$ then $f$ has a local maximum at $x = c$.
- If $f''(c) = 0$ then the test is inconclusive. $f$ may or may not have a local extremum at $x = c$.
  (In this case, we would appeal to the first derivative test.)

**NOTE:** With Taylor Polynomials, we can appreciate this a bit more. If $f'(c) = 0$, then 'near' $x = c$:

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 = f(c) + \frac{1}{2}f''(c)(x-c)^2$$

Hence, locally, the graph of $y = f(x)$ is a parabola with leading coefficient $\frac{1}{2}f''(c)$.

So if $f''(c) > 0$, the graph is locally a 'happy' parabola while if $f''(c) < 0$, the graph is locally a 'sad' parabola.

## EXTENDING TO FUNCTIONS OF TWO VARIABLES

**DEFINITION:** Suppose $f$ is defined on a region $R$:

- The value $f(a, b)$ is called a **local maximum** if $f(a, b) \geq f(x, y)$ for all $(x, y)$ in some open disk $(a, b)$.
- The value $f(a, b)$ is called a **local minimum** if $f(a, b) \leq f(x, y)$ for all $(x, y)$ in some open disk $(a, b)$.

**FERMAT'S THEOREM:** If $f(a, b)$ is a local extreme value of $f$, then $\nabla f(a, b) = \vec{0}$ or $\nabla f(a, b)$ does not exist.

**DEFINITION:** A point $(a, b)$ in the domain of $f$ is a **critical point** if $\nabla f(a, b) = \vec{0}$ or $\nabla f(a, b)$ does not exist.

**NOTE:**

$\nabla f(a, b) = \vec{0}$ means $f_x(a, b) = 0$ **and** $f_y(a, b) = 0$

$\nabla f(a, b)$ does not exist means either $f_x(a, b)$ **or** $f_y(a, b)$ (or both!) does not exist.

**EXAMPLE 1:** Find the critical points of the given function.

1. $f(x, y) = x^2 - 4x + y^2 - xy$

Ans: $\left(\dfrac{8}{3}, \dfrac{4}{3}\right)$

2. $f(x, y) = 2 + 4xy - x^4 - y^4$

Ans: $(-1, -1)$, $(0, 0)$, $(1, 1)$

**EXAMPLE 2:** Find the critical points of the given function.

1. $f(x, y) = x^3 - 3xy^2$

Ans: $(0, 0)$

2. $f(x, y) = x^2 - 2xy + y^2$

Ans: any point on the line $y = x$: $\{(x, y) \, : \, y = x\}$

3. $f(x, y) = 3xe^y - x^3 - e^{3y}$

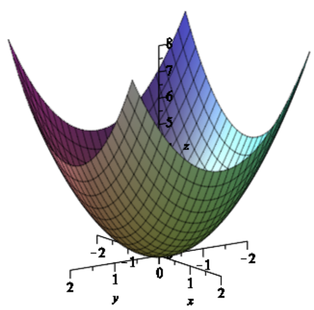
Ans: $(1, 0)$.

**THE SECOND PARTIALS TEST FOR LOCAL EXTREMA:**

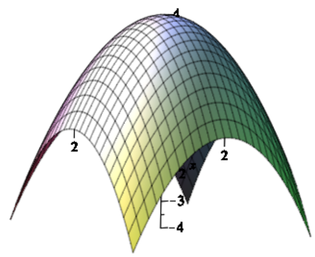
Suppose $f$ has continuous second partials in an open disk containing $(a,b)$ and $\nabla f(a,b) = \vec{0}$.

Let $d = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$

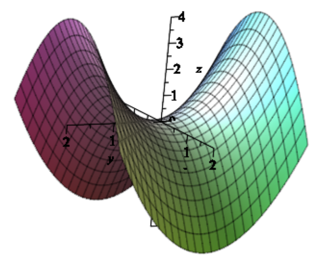
- If $d > 0$
  - and $f_{xx}(a,b) > 0$ then $f$ has a local **minimum** at $(a,b,f(a,b))$.
  - and $f_{xx}(a,b) < 0$ then $f$ has a local **maximum** at $(a,b,f(a,b))$.
- If $d < 0$ then $f$ has a **saddle point** at $(a,b,f(a,b))$.
- If $d = 0$ then the test is inconclusive. (We need more information to determine what is going on.)



Local Minimum



Local Maximum



Saddle Point

**QUESTION:** What can you say about $f_{xx}(a,b)$, $f_{yy}(a,b)$ and $f_{xy}(a,b)$ in the situation that:

- $d > 0$?

- $d < 0$?

**NOTE:** This is proved using Taylor Polynomials... in two dimensions! 'Near' $(a,b)$:

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2$$

If $\nabla f(a,b) = \vec{0}$, then:

$$f(x,y) \approx f(a,b) + \frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2$$

The second partials test is proved by looking at surfaces of the form $z = Ax^2 + Bxy + Cy^2$ and the relationships between the coefficients $A$, $B$, and $C$ which produce minimums, maximums, and saddle points.

**EXAMPLE 3:** Use the second partials test to find and classify the relative extrema for each function.

Check your answer using a graphing utility.

1. $f(x, y) = x^2 - 4x + y^2 - xy$

Ans: Local minimum at $\left(\frac{8}{3}, \frac{4}{3}, -\frac{16}{3}\right)$

2. $f(x, y) = 2 + 4xy - x^4 - y^4$

Ans: Local maximums at $(-1, -1, 4)$ and $(1, 1, 4)$; saddle point at $(0, 0, 2)$.

3. $f(x, y) = 3xe^y - x^3 - e^{3y}$

Ans: Local maximum at $(1, 0, 1)$

**EXAMPLE 4:** Show the second partials test is inconclusive for the critical points each of the functions below.

Determine the behavior near the critical points with help from a graphing utility.

1. $f(x, y) = x^3 - 3xy^2$

Ans: Saddle point at $(0, 0, 0)$.

**NOTE:** This surface is called a 'monkey saddle.' Can you see why?

2. $f(x, y) = x^2 - 2xy + y^2$

All points on the line $y = x$ are local minimums: $\{(x, y, 0) \, : \, y = x\}$

# ABSOLUTE EXTREMA

**RECALL:** Suppose $f$ is defined on an interval $I$.

- The value $f(c)$ is the **absolute (global) maximum** of $f$ on $I$ if $f(c) \geq f(x)$ for all $x$ in $I$.
- The value $f(c)$ is the **absolute (global) minimum** of $f$ on $I$ if $f(c) \leq f(x)$ for all $x$ in $I$.

**NOTE:** The maximum and minimum values of a function are called the '**extreme values**' or the '**extrema**.' Geometrically, the absolute extrema of function correspond to the highest and lowest points on the graph.

**EXTREME VALUE THEOREM:** (EVT) If $f$ is **continuous** on $[a, b]$, then $f$ attains its extrema on $[a, b]$.

That is, there are values $x_1$ and $x_2$ in $[a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x$ in $[a, b]$.

## OPTIMIZING CONTINUOUS FUNCTIONS ON CLOSED BOUNDED INTERVALS:

1. Verify $f$ is continuous on $[a, b]$, so the EVT applies.
2. Find the critical numbers of $f$ in $(a, b)$: solve $f'(x) = 0$ or determine where $f'(x)$ does not exist.
3. Evaluate $f$ at the endpoints and the critical values:
   - the largest function value is the absolute maximum of $f$.
   - the smallest function value is the absolute minimum of $f$.

## GENERALIZING TO FUNCTIONS OF TWO VARIABLES:

**DEFINTION:** Suppose $f$ is defined on a region $R$.

- The value $f(a, b)$ is the **absolute (global) maximum** of $f$ on $R$ if $f(a, b) \geq f(x, y)$ for all $(x, y)$ in $R$.
- The value $f(a, b)$ is called the **absolute (global) minimum** of $f$ on $R$ if $f(a, b) \leq f(x, y)$ for all $x$ in $R$.

**EVT:** If $f$ is **continuous** on a **closed**, **bounded** region $R$, then $f$ attains its extrema on $R$.

That is, there are points $(x_1, y_1)$ and $(x_2, y_2)$ in $R$ such that $f(x_1, y_1) \leq f(x, y) \leq f(x_2, y_2)$ for all $(x, y)$ in $R$.

## OPTIMIZING CONTINUOUS FUNCTIONS ON CLOSED BOUNDED REGIONS:

1. Verify $f$ is continuous on $R$, so the EVT applies.
2. Find the critical points of $f$ in the **interior** of $R$: find where $\nabla f(x, y) = \vec{0}$ or where $\nabla f(x, y)$ d.n.e.
3. Optimize $f$ on the boundary of $R$ using Calculus 1 techniques.
4. Compare the maximum and minimum of $f$ on the boundary with the function values at the critical points.

**EXAMPLE 5:** Let $f(x, y) = 4x + 3y - x^2 - y^2$ and let $R$ be the region $\{(x, y) : 0 \leq x \leq 4,\ 0 \leq y \leq x\}$.

Optimize (that is, find the absolute maximum and minimum) of $f$ over $R$.

Ans: absolute maximum of $\frac{25}{4} = 6.25$ at $\left(2, \frac{3}{2}\right) = (2, 1.5)$; absolute minimum $-4$ at $(4, 4)$.

**RECALL:**

**LONE CRITICAL VALUE**: If $f$ is **continuous** on an interval $I$ and $c$ is the *only* critical value of $f$ in $I$:

- if $f(c)$ is a local maximum value, then $f(c)$ is the absolute maximum on $I$.
- if $f(c)$ is a local minimum value, then $f(c)$ is the absolute minimum on $I$.

**EXAMPLE 6:** (There is no 'Lone Critical **Point**' Theorem.) Consider $f(x, y) = 3xe^y - x^3 - e^{3y}$.

You showed earlier that $(1, 0)$ is the only critical point of $f$ and that critical point results in a local maximum. Find $f(1, 0)$ and $f(-3, 0)$ to prove $f(1, 0)$ is not an absolute maximum. Graph $f$ using a graphing utility.

## GUIDELINES FOR WORKING APPLIED OPTIMIZATION PROBLEMS

1. Identify the quantity you wish to maximize or minimize. (This is often called the 'objective' function.)
2. Assign variables, as needed, and use formulas given in the problem (or from geometry) to relate the variables to the objective function. It is helpful to carry units along here to make sure the equations make sense at the 'level of units.' For example, you can't add feet to inches and get square feet.
3. Since this Calc 3, you will need to get your objective function as a function of just *two* variables. You may need to use some of the formulas from Step 2 to do this.
4. Determine a reasonable applied domain for the problem. For example, if you're asked to build a fence, the length and width of the fence need to be positive quantities.
5. Since there is no 'Lone Critical Point' property for functions of two variables, aim to determine reasonable real-world constraints which result in the applied domain being a closed and bounded region.
6. Check the reasonableness of your answer. If your minimum 'area' is $-300$ square feet, you either did something really wrong, or have just found something to publish in the Journal of Theoretical Physics.
7. Don't give up! Sometimes you need to abandon one line of thinking completely and start from scratch.

**EXAMPLE 7:** The profit in **thousands** of dollars from selling $x$ **hundred** phones and $y$ **hundred** tablets is:

$$P(x,y) = -x^2 - y^2 + 20x + 10y - 25, \quad x, y \geq 0$$

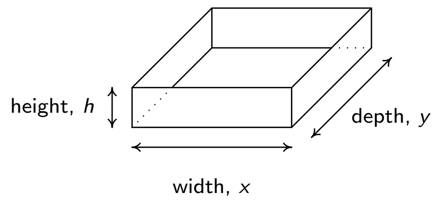
Find the maximum profit and how many phones and tablets should be sold to achieve the maximum profit.

**NOTE:** We have that $x \geq 0$ and $y \geq 0$. What other reasonable condition(s) can we place on the domain?

**HINT:** If we're maximizing profit, then it stands to reason we want a **positive** profit.

Ans: Maximum profit is $100,000 when selling 1000 phones and 500 tablets.

**EXAMPLE 8:** A box with no top is constructed from material which costs \$1 per square foot for the sides and \$1.50 per square foot for the base. For a cost of \$25, find the dimensions of the box which maximize volume.




1. Write an equation for the cost to make the box in terms of $x$, $y$, and $h$ and solve for $h$ in terms of $x$ and $y$.

   **HINT:** Remember - the box has no top!

Ans: $2xh + 2yh + 1.5xy = 25$, so $h = \dfrac{25 - 1.5xy}{2x + 2y}$

2. Use the equation for cost to help you find an equation for the volume $V$ as a function of $x$ and $y$, $V(x, y)$.

Ans: $V = xyh$ so $V(x, y) = \dfrac{xy\,(25 - 1.5xy)}{2x + 2y}$

3. Show that the 'usual' restrictions: $x, y, h > 0$ do not produce a closed, bounded region.

4. Find the critical point(s) of the function $V(x, y)$ and the resulting volume at the critical point(s).

Ans: $V_x(x, y) = \dfrac{y^2\left(50 - 6xy - 3x^2\right)}{(2x + 2y)^2}$ and $V_y(x, y) = \dfrac{x^2\left(50 - 6xy - 3y^2\right)}{(2x + 2y)^2}$

Critical point: $(x, y) = \left(\dfrac{5\sqrt{2}}{3}, \dfrac{5\sqrt{2}}{3}\right) \approx (2.36, 2.36)$, $V\left(\dfrac{5\sqrt{2}}{3}, \dfrac{5\sqrt{2}}{3}\right) = \dfrac{125\sqrt{2}}{18} \approx 9.82\,\text{ft}^3$

5. How can you show the volume you think is the maximum volume really is the maximum volume?

**EXAMPLE 9:** Repeat the last example but instead of the cost being \$25 per box, suppose it is $\$C$ per box.

$V(x,y) = \dfrac{xy\,(C - 1.5xy)}{2x+2y}$, critical point: $(x,y) = \left(\dfrac{\sqrt{2C}}{3}, \dfrac{\sqrt{2C}}{3}\right)$, $V\left(\dfrac{\sqrt{2C}}{3}, \dfrac{\sqrt{2C}}{3}\right) = \dfrac{C\sqrt{2C}}{18}\text{ft}^3$

**HOMEWORK:** Section 15.7: 9 - 65 every other odd, 74 - 76*